

HILBERT C^* -MODULES FROM GROUP ACTIONS: BEYOND THE FINITE ORBITS CASE

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ABSTRACT. Continuous actions of topological groups on compact Hausdorff spaces X are investigated which induce almost periodic functions in the corresponding commutative C^* -algebra. The unique invariant mean on the group resulting from averaging allows to derive a C^* -valued inner product and a Hilbert C^* -module which serve as an environment to describe characteristics of the group action. For uniformly continuous, Lyapunov stable actions the derived invariant mean $M(\phi_x)$ is continuous on X for any element $\phi \in C(X)$, and the induced C^* -valued inner product corresponds to a conditional expectation from $C(X)$ onto the fixed point algebra of the action defined by averaging on orbits. In the case of selfduality of the Hilbert C^* -module all orbits are shown to have the same cardinality. Stable actions on compact metric spaces give rise to C^* -reflexive Hilbert C^* -modules. The same is true if the cardinality of finite orbits is uniformly bounded and the number of closures of infinite orbits is finite. A number of examples illustrate typical situations appearing beyond the classified cases.

1. INTRODUCTION

Investigating continuous group actions on topological spaces several mathematical approaches may be applied. In the present paper the authors continue their work started in [5, 18] which relies on the Gel'fand duality of locally compact Hausdorff spaces and commutative C^* -algebras. In the dual picture some well-known results from functional analysis and noncommutative geometry can be applied to get new insights, often also for related noncommutative situations of group actions on general C^* -algebras.

Consider a continuous action of a topological group Γ on a compact Hausdorff space X . Following the Gel'fand duality it can be seen as a continuous action of Γ on the commutative C^* -algebra $C(X)$ of all continuous complex-valued functions on X . Let us denote the subalgebra of Γ -invariant functions on X by $C_\Gamma(X) \subset C(X)$.

We wish to introduce the structure of a pre-Hilbert C^* -module over $C_\Gamma(X)$ on $C(X)$ which expresses significant properties of the action of Γ on X . One way to find suitable C^* -valued inner products on $C(X)$ is the search for conditional expectations $E : C(X) \rightarrow C_\Gamma(X)$ which are a kind of mean over the group action of Γ on $C(X)$ and canonically give rise to the Hilbert $C_\Gamma(X)$ -module structures on $C(X)$ we are looking for. We followed that approach in [5, 18] (see [15] for a related discussion).

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Here we want to consider a more general approach closer to the topological background. For elements $\phi, \psi \in C(X)$ and for the derived group maps

$$(1) \quad \phi_x : \Gamma \rightarrow \mathbb{C}, \quad \phi_x(g) = \phi(gx), \quad (x \in X)$$

we want to select a suitable normalized invariant mean m_Γ on Γ such that a $C_\Gamma(X)$ -valued inner product on $C(X)$ could be defined like

$$(2) \quad \langle \phi, \psi \rangle(x) := m_\Gamma(\phi_x \overline{\psi_x}), \quad (x \in X).$$

Of course, we would have to suppose Γ to be amenable at this point to warrant the existence of the (left) invariant mean m_Γ . The product (2) has to satisfy at least the following two properties (with Γ -invariance following from the definition):

- 1) The resulting functions $m_\Gamma(\phi_x \overline{\psi_x})$ are continuous in the argument $x \in X$.
- 2) The value $\langle \phi, \phi \rangle(x)$ is always positive, if $\phi(gx) \neq 0$ for some $g \in \Gamma$, some $x \in X$.

One can observe that the property 2) would e.g. follow from the following supposition:

- 2') For any $x \in X$ and any non-zero $\phi \in C(X)$ the map ϕ_x is a non-zero almost periodic function on Γ .

The supposition 2') would allow us:

- to avoid the restriction on Γ to be amenable,
- to overcome the dependence on the particular choice of m_Γ ,

by passing from (2) to

$$(3) \quad \langle \phi, \psi \rangle(x) := M(\phi_x \overline{\psi_x}),$$

where the map $M : \Gamma \rightarrow \mathbb{C}$ is the unique invariant mean on almost periodic functions with respect to the given action of Γ , when 1) and 2') are supposed to hold (cf. the appendix). The link to results in [5, 18] is given by constructing a suitable conditional expectation $E_\Gamma : C(X) \rightarrow C_\Gamma(X)$ by the rule

- 1') For any $\varphi \in C(X)$ the function $E_\Gamma(\varphi)(x) := M(\varphi_x)$ is continuous in x .

Properties 1') and 2') provide that the formula (2) makes $C(X)$ a pre-Hilbert C^* -module over $C_\Gamma(X)$. Let us denote its completion by $L_\Gamma(X)$.

We are interested in two questions here:

- (1) For which actions the conditions 1') and 2') hold?
- (2) If they hold, what properties does $L_\Gamma(X)$ have?

Our reference on almost periodic functions is [3]. Hilbert C^* -modules were introduced in [14] and [10]. For facts on Hilbert C^* -modules we refer the reader to [8, 7, 13]. Recall that for a Hilbert C^* -module L over a C^* -algebra A the A -dual module L' is the module of all bounded A -linear maps from L to A . L is called self-dual (resp. C^* -reflexive) if $L = L'$ (resp. if $L = L''$).

Our paper is organized as follows: In the Section 2 we will give some sufficient conditions for conditions 1') and 2') to hold, hence, for the existence of $C_\Gamma(X)$ -valued inner products on the C^* -algebra $C(X)$. We also show that our type of averaging is the same as the averaging over orbits. Section 3 deals with the more restrictive situations in which the resulting Hilbert C^* -module turns out to be self-dual. In Section 4 we revisit the situation of resulting C^* -reflexive Hilbert C^* -modules and obtain an important restriction on X to be supposed. In Section 5 we give some examples showing different possible behavior of

averaging. The Appendix is devoted to the proof of the uniqueness of a measure used for averaging.

2. LYAPUNOV STABILITY AND CONTINUITY OF AVERAGING

We want to find conditions under which a well-defined averaging over the group action on orbits exists in the case of infinite orbits. For this aim we introduce additionally to the condition of uniform continuity discussed in [5, 18] the condition of Lyapunov stability. The latter condition ensures uniform continuity, the well-definedness of averaging and the existence of a conditional expectation onto the fixed-point algebra which gives rise to a C^* -valued inner product and a resulting Hilbert C^* -module structure. In subsequent sections we can apply this tool to characterize those group actions on compact Hausdorff spaces with infinite orbits.

Definition 2.1. We say that an action of a group G on a locally compact Hausdorff space X is *uniformly continuous* if for every point $x \in X$ and every neighborhood U_x of x there exists a neighborhood V_x of x such that $g(V_x) \subseteq U_x$ for every $g \in G_x$, where G_x denotes the stabilizer of x .

Theorem 2.2. *Let an action of a topological group Γ on a compact Hausdorff space X be uniformly continuous. If all orbits are finite and if their size is uniformly bounded then the average $M(\varphi_x)$ is continuous with respect to $x \in X$ for any $\varphi \in C(X)$.*

Proof. If an orbit Γx is finite then the function φ_x on Γ is exactly periodic, hence

$$M(\varphi_x) = \frac{1}{\#\Gamma x} \sum_{gx \in \Gamma x} \varphi(gx),$$

so the average on Γ is the same as the average over orbits. Continuity of the latter is provided by Lemma 2.11 from [5]. \square

Example 5.2 below demonstrates that in the case of presence of infinite orbits the uniform continuity is not sufficient for the continuity of the average.

Now we generalize an approach of [18] and introduce a condition which is sufficient to overcome these difficulties.

Let Φ be a uniform structure on a compact space X . Recall from [1] that, on a compact space, there is a unique uniform structure compatible with its topology. It consists of *all* neighborhoods of the diagonal in $X \times X$ [1, Ch. II, Sect. 4, Theorem 1]. If X is a metric space with a metric d then the uniform structure is the set of the neighborhoods of the diagonal $\Delta \subset X \times X$ of the form $\{(x, y) : x, y \in X, d(x, y) < \varepsilon\}$, $\varepsilon \in (0, \infty)$.

Definition 2.3. An action of a group Γ on a topological space X with a uniform structure Φ compatible with its topology is called *Lyapunov stable* if for any $\mathbb{U} \in \Phi$ and any $x \in X$ there is $\mathbb{V} \in \Phi$ such that $(gx, gy) \in \mathbb{U}$ for any $g \in \Gamma$ if $(x, y) \in \mathbb{V}$.

Note that in the case of a *metric* space, this definition takes the following form:

Definition 2.4. An action of a group Γ on a metric space X is called *Lyapunov stable* if for any $\varepsilon > 0$ and any $x \in X$ there exist $\delta > 0$ such that

$$\rho(gx, gy) < \varepsilon \quad \text{for any } g \in \Gamma \quad \text{if } \rho(x, y) < \delta.$$

Lemma 2.5. *If an action of a discrete group Γ on a topological space X with a uniform structure is Lyapunov stable then it is uniformly continuous.*

Proof. For $x \in X$ and for $\mathbb{U} \in \Phi$ set $\mathbb{U}(x) := \{y \in X : (x, y) \in \mathbb{U}\}$. If W is a neighborhood of x then there is $\mathbb{U} \in \Phi$ such that $\mathbb{U}(x) \subset W$. By stability, there is $\mathbb{V} \in \Phi$ such that $(gx, gy) \in \mathbb{U}$ for any $g \in \Gamma$ if $(x, y) \in \mathbb{V}$. Now let $g \in \Gamma x$. Take any $y \in \mathbb{V}(x)$. Then $(x, gy) \in \mathbb{U}$, hence $gy \in \mathbb{U}(x) \subset W$, i.e. $g(\mathbb{V}(x)) \subset W$ for any $g \in \Gamma x$. \square

In the case when all orbits are finite, uniform continuity is equivalent to Lyapunov stability:

Proposition 2.6. *Let a discrete group act uniformly continuously on a compact Hausdorff space X and let all the orbits are finite. Then this action is Lyapunov stable.*

Proof. Take a neighborhood \mathbb{W} of the diagonal in $X \times X$ and take a point $x \in X$. Since its orbit is finite, we can take a finite set $\{g_1, \dots, g_s\}$ of elements in Γ such that $\{g_1x, \dots, g_sx\}$ is the orbit Γx . Now find a neighborhood U^x of x such that $g_i(U^x) \times g_i(U^x) \subset \mathbb{W}$ for each $i = 1, \dots, s$.

Uniform continuity implies that there exists a neighborhood V^x of x such that $hy \in U^x$ for any $y \in V^x$ and any $h \in \Gamma x$. Since any $g \in \Gamma$ can be written as $g = g_i h$ for some $i = 1, \dots, s$ and for some $h \in \Gamma x$, $g(V^x) = g_i(h(V^x)) \subset g_i(U^x)$.

It follows from compactness of X that there is a finite number of points x_1, \dots, x_r in X such that the sets V^{x_1}, \dots, V^{x_r} form a finite covering for X . Then $\mathbb{W}_0 = V^{x_1} \times V^{x_1} \cup \dots \cup V^{x_r} \times V^{x_r}$ is a neighborhood of the diagonal in $X \times X$.

Take $(y, z) \in \mathbb{W}_0$. Then there is some $1 \leq j \leq r$ such that $(y, z) \in V^{x_j} \times V^{x_j}$. Then $(gy, gz) \in g_i(U^{x_j}) \times g_i(U^{x_j})$ for some i . By construction, $g_i(U^{x_j}) \times g_i(U^{x_j}) \subset \mathbb{W}$, so we conclude that $(gy, gz) \in \mathbb{W}$ for any $g \in \Gamma$ whenever $(y, z) \in \mathbb{W}_0$. \square

Proposition 2.7. *Let a discrete group Γ act Lyapunov stably on a compact Hausdorff space X and let $\varphi : X \rightarrow \mathbb{C}$ be a continuous function. Then, for any $x \in X$, the function $\varphi_x : \Gamma \rightarrow \mathbb{C}$, $\varphi_x(g) := \varphi(gx)$, is almost periodic.*

Proof. Take $\varepsilon > 0$. Then compactness of X implies existence of some $\mathbb{U} \in \Phi$ such that

$$(4) \quad |\varphi(x) - \varphi(y)| < \varepsilon \quad \text{if } (x, y) \in \mathbb{U},$$

since φ is uniformly continuous and Φ consists of all neighborhoods of the diagonal.

Stability implies that there is $\mathbb{V} \in \Phi$ such that $(gx, gy) \in \mathbb{U}$ for any g if $(x, y) \in \mathbb{V}$.

Denote the closure of the orbit Γx by $Y \subset X$. It is a compact subset. Compactness of Y implies that one can find in Y a finite number of points of the form $g_i x$, $i = 1, \dots, s$, such that for any $p \in Y$ there is some i , for which $(g_i x, p) \in \mathbb{V}$ and, therefore, for any $g, h \in \Gamma$,

$$(5) \quad (hg_i x, hgx) \in \mathbb{U}.$$

Then the functions $L_{g_i} \varphi_x$, $i = 1, \dots, s$, form an ε -net for the set $\{L_g \varphi_x : g \in \Gamma\}$, with respect to the uniform norm. Indeed, for any $g \in \Gamma$, there is an index i such that (5) holds. Then, by (4), we have

$$\sup_{h \in \Gamma} |(L_g \varphi_x)(h) - (L_{g_i} \varphi_x)(h)| = \sup_{h \in \Gamma} |\varphi(hgx) - \varphi(hg_i x)| < \varepsilon.$$

\square

So, under the conditions of Proposition 2.7 the invariant mean $M(\varphi_x)$ is well-defined on $C(X)$.

Theorem 2.8. *Let a discrete group Γ act on a compact Hausdorff space X . If the action is Lyapunov stable, then the conditional expectation $E_\Gamma : C(X) \rightarrow C_\Gamma(X)$ defined by $E_\Gamma(\phi)(x) = M(\phi_x)$ is well-defined, i.e. the conditions 1') and 2') hold.*

Proof. By Proposition 2.7 we only need to verify the continuity of the mean $M(\varphi_x)$ with respect to $x \in X$. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Let us remind (cf. [6, pp. 250–251]) that we can choose $h_1, \dots, h_p \in \Gamma$ in such a way that the uniform distance on $\Gamma \times \Gamma$ between the function

$$\frac{1}{p} \sum_{j=1}^p D_{h_j} \varphi_x : \Gamma \times \Gamma \rightarrow \mathbb{C} \quad (\text{where } (D_h \psi)(g_1, g_2) := \psi(g_1 h g_2))$$

and some constant is less than ε . In this case the uniform distance satisfies the inequality

$$\left\| M(\varphi_x) - \frac{1}{p} \sum_{j=1}^p D_{h_j} \varphi_x \right\|_u < 2\varepsilon.$$

Let us choose a neighborhood $\mathbb{V} \in \Phi$ such that

$$|\varphi(gy) - \varphi(gx)| < \varepsilon, \quad \text{for any } g \in \Gamma, (x, y) \in \mathbb{V}.$$

This neighborhood can be found as in the proof of Proposition 2.7: first we can find a neighborhood \mathbb{U} such that $|\varphi(y) - \varphi(z)| < \varepsilon$ whenever $(y, z) \in \mathbb{U}$ (using compactness of X). Then, by stability of the action, we can find for this \mathbb{U} another neighborhood $\mathbb{V} \in \Phi$ such that $(gy, gx) \in \mathbb{U}$ for any $g \in \Gamma$ whenever $(y, x) \in \mathbb{V}$.

Then for any $y \in \mathbb{U}(x)$ one has

$$\begin{aligned} \left(\frac{1}{p} \sum_{j=1}^p D_{h_j} \varphi_y \right) (g_1, g_2) &= \\ &= \left(\frac{1}{p} \sum_{j=1}^p D_{h_j} \varphi_x \right) (g_1, g_2) + \frac{1}{p} \sum_{j=1}^p (\varphi_y(g_1 h_j g_2) - \varphi_x(g_1 h_j g_2)) \\ &= \left(\frac{1}{p} \sum_{j=1}^p D_{h_j} \varphi_x \right) (g_1, g_2) + \frac{1}{p} \sum_{j=1}^p (\varphi(g_1 h_j g_2 y) - \varphi(g_1 h_j g_2 x)). \end{aligned}$$

Each term of the second summand is less than ε . Hence, the second summand is less than ε . Thus,

$$\left\| M(\varphi_x) - \frac{1}{p} \sum_{j=1}^p D_{h_j} \varphi_y \right\|_u < 3\varepsilon$$

for any $y \in \mathbb{U}(x)$. Therefore, considering $M(\varphi_x)$ as an arbitrary constant, we have

$$\left\| M(\varphi_y) - \frac{1}{p} \sum_{j=1}^p D_{h_j} \varphi_y \right\|_u < 6\varepsilon,$$

and finally,

$$|M(\varphi_y) - M(\varphi_x)| < 9\varepsilon$$

for any $y \in \mathbb{U}(x)$. Consequently, $E_\Gamma(\phi)$ is Γ -invariant and continuous on X . \square

For $x \in X$ let us denote its orbit Γx by γ and the closure of the orbit γ in X by $\overline{\gamma}$.

Theorem 2.9. *Let a discrete group Γ act on a compact Hausdorff space X .*

1) *For a Lyapunov stable action and for the unique invariant mean $M : \Gamma \rightarrow \mathbb{C}$ we have the equality*

$$(6) \quad M(\varphi_x) = \int_{\overline{\gamma}} \varphi|_{\overline{\gamma}} d\mu_{\overline{\gamma}},$$

where $x \in \gamma$, $\mu_{\overline{\gamma}}$ is a (unique) invariant measure on $\overline{\gamma}$ of total mass 1.

2) *If γ is finite, then $M(\varphi_x)$, $x \in \gamma$, can be taken as the standard average, as it was considered in [5, 18].*

Proof. Evidently, 2) follows from 1).

Let us show that for a $\varphi \in C(X)$ the left-hand side of (6) does not depend on $x \in \overline{\gamma}$. First, evidently, it does not depend on the choice of x inside the same orbit. Hence, it is sufficient to verify that the value is the same for gx_2 sufficiently close to any x_1 for $x_1, x_2 \in \overline{\gamma}$ to demonstrate the invariantness with respect to the action of Γ . By the Lyapunov stability property, for any $\varepsilon > 0$ we can find an element $g_\varepsilon \in \Gamma$ such that $g_\varepsilon x_2$ is so close to x_1 that $|\varphi(gx_1) - \varphi(gg_\varepsilon x_2)| < \varepsilon$ for any $g \in \Gamma$. Then

$$\begin{aligned} |M(\varphi_{x_1}) - M(\varphi_{x_2})| &= |M(\varphi_{x_1}) - M(\varphi_{g_\varepsilon x_2})| = |M(\varphi_{x_1} - \varphi_{g_\varepsilon x_2})| \\ &\leq \sup_{g \in \Gamma} |\varphi(gx_1) - \varphi(gg_\varepsilon x_2)| < \varepsilon. \end{aligned}$$

Since ε is arbitrary small, $M(\varphi_{x_1}) = M(\varphi_{x_2})$ and the value is constant on the closure of orbits.

A similar estimation implies the continuity of this (well-defined by the above argument) functional $m : C(\overline{\gamma}) \rightarrow \mathbb{C}$, $m(\phi) = M(\phi_x)$ for $x \in \overline{\gamma}$, with respect to the variation of closures of orbits. By the Riesz-Markov-Kakutani theorem [4, Theorem 3, Sect. IV.6], m has the form

$$m(f) = \int_{\overline{\gamma}} f d\mu,$$

where μ is some regular countably additive complex measure on $\overline{\gamma}$. Evidently, μ is invariant. It remains to explain why μ is unique. In fact, this follows from [2, Ch. VII, § 1, Problem 14]. We give details in the Appendix. \square

3. SELF-DUALITY

After a characterization of the inner structure of Hilbert C^* -modules that arise from Lyapunov stable actions we are going to describe the interrelation between certain properties of the action and self-duality of the resulting Hilbert C^* -module.

Lemma 3.1. *Let a discrete group Γ act on a compact Hausdorff space X . If the action is Lyapunov stable then any two orbits are either separated from each other, or have the same closure. Thus, closures of orbits are separated sets in X/Γ . The Gelfand spectrum of $C_\Gamma(X)$ is the set of closures of orbits.*

Proof. Suppose, two orbits $\gamma = \Gamma x$ and $\gamma' = \Gamma y$ are not separated but have distinct closures. This means (after a shift, if necessary) that $y \in \overline{\gamma}$, but $g_0 y \notin \overline{\gamma}$ for some $g_0 \in \Gamma$. Then there exists some \mathbb{U} of the uniform structure Φ such that there are no points of the form $(g_0 y, g_1 x)$ in \mathbb{U} . Let us take a neighborhood $\mathbb{V} \in \Phi$ corresponding to \mathbb{U} by the definition of Lyapunov stability. Take $(y, g_2 x) \in \mathbb{V}$. Then $(g_0 y, g_0 g_2 x) \in \mathbb{U}$. Take $g_1 = g_0 g_2$. A contradiction.

Thus, the quotient space of closures of orbits is Hausdorff and, hence, it coincides with the Gelfand spectrum of $C_\Gamma(X)$. \square

Theorem 3.2. *Let a discrete group Γ act on a compact Hausdorff space X . In the case of a Lyapunov stable action the module $L_\Gamma(X)$ has the following description: it consists of all functions $\psi : X \rightarrow \mathbb{C}$ such that*

- 1) $\psi|_{\overline{\gamma}} \in L_2(\overline{\gamma}, \mu_{\overline{\gamma}})$, where $\mu_{\overline{\gamma}}$ is a unique normalized invariant measure on $\overline{\gamma}$ for any orbit γ ,
- 2) for any $\varphi \in C(X)$ the function $\langle \psi, \varphi \rangle_L$ is continuous.

In particular, the average $\langle \psi, \mathbf{1} \rangle_L$ of such a function ψ is continuous on X/Γ .

Proof. We should prove the following two facts: a) the set of continuous functions on X satisfies these conditions, and b) it is dense in this set with respect to the C^* -valued inner product on the module $L_\Gamma(X)$.

Condition 1) of the assertion above should be interpreted via the equality

$$(7) \quad \int_{\overline{\gamma}} \psi|_{\overline{\gamma}} d\mu_{\overline{\gamma}} = M(\psi_x), \quad \psi \in C(X), \quad x \in \gamma.$$

This equality follows from two facts: i) the left-hand side depends only on $\psi|_{\overline{\gamma}}$ and defines an invariant mean on almost periodic functions on Γ , ii) such a mean is unique. (see Theorem 2.9)

Thus, by the results of Proposition 2.7 and Theorem 2.8 of Section 2, condition a) is fulfilled.

Now take an arbitrary function $\psi(x)$ satisfying the conditions 1) and 2) of the assertion above, an arbitrary function $\varphi \in C(X)$ with $\|\varphi\|_L \leq 1$, and an arbitrary small $\varepsilon > 0$. Consider the closure of an orbit $\overline{\gamma}$. Choose a continuous function $f_{\overline{\gamma}} : \overline{\gamma} \rightarrow \mathbb{C}$ such that

$$(8) \quad \int_{\overline{\gamma}} |\psi|_{\overline{\gamma}} - f_{\overline{\gamma}}|^2 d\mu < \varepsilon^2.$$

By normality of X , $f_{\overline{\gamma}}$ can be extended to a continuous function $\widehat{f}_{\overline{\gamma}} : X \rightarrow \mathbb{C}$. There exists a neighborhood $U_{\overline{\gamma}}$ of $\overline{\gamma}$ in the Gelfand spectrum $\widetilde{X/\Gamma}$ of $C_\Gamma(X)$, such that

$$(9) \quad \int_{\overline{\gamma}''} (\psi|_{\overline{\gamma}''} - \widehat{f}_{\overline{\gamma}}|_{\overline{\gamma}''}) \overline{\varphi}_{\overline{\gamma}''} d\mu < 2\varepsilon, \quad \overline{\gamma}'' \in U_{\overline{\gamma}}.$$

This follows from (8) and 2). Choose a finite subcovering $U_{\overline{\gamma}_i}$ of $\widetilde{X/\Gamma}$ and a subordinated partition of unity $\omega_i, i = 1, \dots, I$. This can be done by Lemma 3.1. Then $\sup |\langle \psi - f, \varphi \rangle| \leq 2\varepsilon$, where

$$f = \sum_{i=1}^I \omega_i \widehat{f}_{\overline{\gamma}_i}.$$

Thus $f \in C(X)$ is 2ε -close to ψ . \square

Theorem 3.3. *Let a discrete group Γ act on a compact Hausdorff space X .*

- 1) *Suppose, the module $L_\Gamma(X)$ is self-dual and the Gelfand spectrum $\widetilde{X/\Gamma}$ of the algebra of continuous invariant functions $C_\Gamma(X)$ has no isolated points. Then there are only finitely many $\overline{\gamma}$ with infinite γ and all finite orbits have the same cardinality.*
- 2) *If there are only finitely many $\overline{\gamma}$ with infinite γ and all finite orbits have the same cardinality, then the module $L_\Gamma(X)$ is self-dual.*

Proof. 1) By [12], the restriction on the Gelfand spectrum implies that $L_\Gamma(X)$ is finitely generated projective. Let N be the cardinality of some of its generator systems. Thus, the number of points of each finite orbit is $\leq N$. This follows from the epimorphy of the restriction map $L_\Gamma(X) \rightarrow L_\Gamma(Y)$, where $Y \subset X$ is a closed Γ -invariant set. Indeed, Y is a closed set in a normal space, hence continuous functions on it are extendable by the Tietze theorem.

In this situation of uniform boundness of the cardinality of finite orbits, the subset X_f , formed by all finite orbits, is a closed (invariant) subset of X . Indeed, suppose, an infinite orbit γ is in the closure of X_f . Choose a cover of X by (a finite number of) open sets U_i , such that no one of them is covered by the others, and γ is covered by more than N of these U_i 's. Let $\mathbb{U} = \cup_i (U_i \times U_i)$ be an element of the uniform structure on X . Then there exists another neighborhood \mathbb{V} of the diagonal in $X \times X$ such that $\Gamma(\mathbb{V}) \subset \mathbb{U}$ under the diagonal action. Choosing a finite orbit (of cardinality $\leq N$) \mathbb{V} -close to γ we obtain a contradiction to properties of \mathbb{U} .

Thus, as above, $L_\Gamma(X_f)$ is finitely generated. Moreover, it is projective, because the projection associated with a canonical isometric embedding of the finitely generated projective $C(\widetilde{X/\Gamma})$ -module $L_\Gamma(X)$ into a standard finitely generated $C(\widetilde{X/\Gamma})$ -module $C(\widetilde{X/\Gamma})^N$, say $\pi : C(\widetilde{X/\Gamma})^N \rightarrow L_\Gamma(X)$, restricted to X_f gives an epimorphic idempotent mapping

$$\pi' : C(\widetilde{X_f/\Gamma})^N \rightarrow L_\Gamma(X_f),$$

defined by the restriction of matrix elements of the projection π . Epimorphy follows from the above argument which relied on the Tietze theorem.

We arrive to the case considered in [5] and [18]. As it is explained in Theorem 2.9, the average over finite orbits is the same as in these papers, and the inner product is the same. Thus, $L_\Gamma(X_f) = C(X_f)$. By the results of [5] and [18], under our assumptions this module is finitely generated projective if and only if all (finite) orbits have the same cardinality.

Now we pass to proving the statement about infinite orbits. Suppose there exists an infinite number of closures of infinite orbits: $\overline{\gamma}_i$, $i = 1, 2, \dots$. We need to construct a $C_\Gamma(X)$ -functional on $L_\Gamma(X)$, which is not an element of $L_\Gamma(X)$.

Passing to a subsequence, if necessary, we can assume that for each point $z_i \in \widetilde{X/\Gamma}$ representing $\overline{\gamma}_i$, we can choose an open neighborhood U_i , such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Indeed, suppose the opposite. This implies that for one of the points, say z_1 , and any its neighborhood only finitely many points from the set $\{z_1, z_2, \dots\}$ are off this neighborhood (i.e., $z_2, z_3, \dots \rightarrow z_0$). We choose a neighborhood $U'_1 \ni z_0$, such that there is $z_{i_1} \notin U'_1$, and (by normality) a neighborhood U''_1 of z_0 , such that $\overline{U''_1} \subset U'_1$ and there is a neighborhood U_1 of z_{i_1} , such that $U_1 \cap \overline{U''_1} = \emptyset$. Take $U'_2 \subset U''_1$, such that there exists $z_{i_2} \in U''_1$, $z_{i_2} \notin U'_2$. Take, by normality, $U''_2 \ni z_{i_0}$, such that $U''_2 \subset \overline{U''_1} \subset U'_1$ and there exists a neighborhood

U_2 of z_{i_2} such that $U_2 \subset U_1''$ and $U_2 \cap \overline{U_2''} = \emptyset$. And so on. Finally, collect the points $\{z_{i_n}\}$ which have the required property.

Let us define $f_i : \overline{\gamma}_i \rightarrow \{0, \sqrt{i}\}$ to be the indicator functions of subsets of $\overline{\gamma}_i$ with $\mu_i(\text{supp } f_i) = \frac{1}{i}$ (where μ_i is the invariant measure on $\overline{\gamma}_i$ of total mass 1). Thus $f_i \in L^2(\overline{\gamma}_i, \mu_i)$, $\langle f_i, f_i \rangle_{\overline{\gamma}_i} = \int_{\overline{\gamma}_i} |f_i|^2 d\mu_i = 1$ and $\int_{\overline{\gamma}_i} |f_i| d\mu_i = 1/\sqrt{i}$.

Choose $\alpha_i \in C(X)$ ($i = 1, 2, \dots$) such that

- 1) $\text{supp } \alpha_i \subset p^{-1}(U_i)$, where $p : X \rightarrow \widetilde{X}/\Gamma$ is the canonical projection;
- 2) $\alpha_i(X) \subset \left[0, \frac{1}{\sqrt{i}}\right]$, $\alpha_i(\overline{\gamma}_i) = \left[0, \frac{1}{\sqrt{i}}\right]$;
- 3) $\|\alpha_i|_{\overline{\gamma}_i} - f_i\|_{L^2} < \frac{1}{2^i}$, $\|\alpha_i|_{\overline{\gamma}_i}\|_{L^2}^2 - \|f_i\|_{L^2}^2 < \frac{1}{2^i}$;
- 4) $\int_{\overline{\gamma}} |\alpha_i|_{\overline{\gamma}}| d\mu_{\overline{\gamma}} \leq \frac{1}{\sqrt{i}} + \frac{1}{2^{i-1}}$ for any $\overline{\gamma}$;
- 5) $\int_{\overline{\gamma}} |\alpha_i|_{\overline{\gamma}}|^2 d\mu_{\overline{\gamma}} \leq 1 + \frac{1}{2^{i-1}}$ for any $\overline{\gamma}$.

To construct these functions we first approximate f_i by an appropriate continuous function, then extend by the Tietze theorem, and finally multiply by an appropriate partition of unity function. More precisely, we first choose a continuous function $\alpha'_i : \overline{\gamma}_i \rightarrow [0, 1/\sqrt{i}]$, restricted to satisfy properties 2 and 3. Then we extend by the Tietze theorem α'_i to a continuous function $\alpha''_i : X \rightarrow [0, 1/\sqrt{i}]$. By Theorem 3.2 the functions

$$\langle \alpha''_i, 1 \rangle_L : \widetilde{X}/\Gamma \rightarrow [0, +\infty), \quad \langle \alpha''_i, \alpha''_i \rangle_L : \widetilde{X}/\Gamma \rightarrow [0, +\infty),$$

are continuous and

$$\langle \alpha''_i, 1 \rangle_L(z_i) \in \left(\frac{1}{\sqrt{i}} - \frac{1}{2^i}, \frac{1}{\sqrt{i}} + \frac{1}{2^i}\right), \quad \langle \alpha''_i, \alpha''_i \rangle_L(z_i) \in \left(1 - \frac{1}{2^i}, 1 + \frac{1}{2^i}\right).$$

Choose a neighborhood $U'_i \subset U_i$ of z_i such that

$$\langle \alpha''_i, 1 \rangle_L(z_i) \in \left(\frac{1}{\sqrt{i}} - \frac{1}{2^{i-1}}, \frac{1}{\sqrt{i}} + \frac{1}{2^{i-1}}\right), \quad \langle \alpha''_i, \alpha''_i \rangle_L(z_i) \in \left(1 - \frac{1}{2^{i-1}}, 1 + \frac{1}{2^{i-1}}\right).$$

Let $\omega_i : \widetilde{X}/\Gamma \rightarrow [0, 1]$ be a continuous function with $\omega_i(z_i) = 1$ and $\text{supp } \omega_i \in U'_i$. Put $\widehat{\omega}_i := p^* \omega_i : X \rightarrow [0, 1]$ and $\alpha_i := \widehat{\omega}_i \alpha''_i$. They are the required ones.

Define a function $h : X \rightarrow [0, +\infty)$ to be equal to α_i on $p^{-1}(U_i)$ ($i = 1, 2, \dots$) and 0 otherwise. First, we wish to show that $h \notin L_\Gamma(X)$. Indeed, $\langle h, h \rangle_L$ is greater than $1 - \frac{1}{2^i} > 1/2$ at each z_i and vanishes at any accumulation point of $\{z_i\}$.

Now let us show that $h \in L_\Gamma(X)'$. Let φ be a continuous function on Y such that $\|\langle \varphi, \varphi \rangle\|_L \leq 1$. Then for any $\overline{\gamma}$ in some $p^{-1}(U_i)$ we have (using property 5)

$$|\langle h, \varphi \rangle_{\overline{\gamma}}| = |\langle \alpha_i|_{\overline{\gamma}}, \varphi|_{\overline{\gamma}} \rangle| \leq \langle \alpha_i|_{\overline{\gamma}}, \alpha_i|_{\overline{\gamma}} \rangle^{1/2} \cdot \langle \varphi, \varphi \rangle^{1/2} \leq 2.$$

For the remaining $\overline{\gamma}$'s this product vanishes. It remains to show that $\langle h, \varphi \rangle$ is a continuous (invariant) function, i.e. that for any $\varepsilon > 0$ and any point $\overline{\gamma}_0$ from the closure of $\cup_i p^{-1}(U_i)$ there is an invariant neighborhood W of $\overline{\gamma}_0$ such that

$$\int_{\overline{\gamma}} \overline{h_{\overline{\gamma}}} \cdot \varphi|_{\overline{\gamma}} d\mu_{\overline{\gamma}} < \varepsilon$$

for any $\overline{\gamma} \in W$. Choose W not intersecting with $p^{-1}(U_i)$ for $i = 1, \dots, k$, where $k > \max \left(2, \left(\frac{2 \sup_{x \in X} |\varphi(x)|}{\varepsilon} \right)^2 \right)$. Then (beyond the trivial cases) $\overline{\gamma} \in p^{-1}(U_i)$, $i > k$. Let us

estimate using property 4):

$$\begin{aligned} \left| \int_{\bar{\gamma}} \overline{h_{\bar{\gamma}}} \cdot \varphi|_{\bar{\gamma}} d\mu_{\bar{\gamma}} \right| &\leq \sup_{x \in X} |\varphi(x)| \cdot \int_{\bar{\gamma}} |\alpha_i|_{\bar{\gamma}} d\mu_{\bar{\gamma}} = \sup_{x \in X} |\varphi(x)| \cdot \left(\frac{1}{\sqrt{i}} - \frac{1}{2^{i-1}}, \frac{1}{\sqrt{i}} + \frac{1}{2^{i-1}} \right) \\ &< \sup_{x \in X} |\varphi(x)| \cdot \frac{2}{\sqrt{i}} < \varepsilon \end{aligned}$$

for $i > k$. Hence, the module is not self-dual.

2) As it is explained in the first part of the proof, in this case

$$\begin{aligned} X &= X_f \sqcup \bar{\gamma}_1 \sqcup \cdots \sqcup \bar{\gamma}_n, \\ L_{\Gamma}(X) &= L_{\Gamma}(X_f) \oplus L^2(\bar{\gamma}_1, \mu_{\bar{\gamma}_1}) \oplus L^2(\bar{\gamma}_n, \mu_{\bar{\gamma}_n}), \\ (L_{\Gamma}(X))'_{C_{\Gamma}(X)} &= (L_{\Gamma}(X_f))'_{C_{\Gamma}(X_f)} \oplus (L^2(\bar{\gamma}_1, \mu_{\bar{\gamma}_1}))'_{C_{\Gamma}(\bar{\gamma}_1)} \oplus (L^2(\bar{\gamma}_n, \mu_{\bar{\gamma}_n}))'_{C_{\Gamma}(\bar{\gamma}_n)} \\ &= (L_{\Gamma}(X_f))'_{C_{\Gamma}(X_f)} \oplus (l^2(\mathbb{C}))'_{\mathbb{C}} \oplus \cdots \oplus (l^2(\mathbb{C}))'_{\mathbb{C}} \\ &= (L_{\Gamma}(X_f))'_{C_{\Gamma}(X_f)} \oplus L^2(\bar{\gamma}_1, \mu_{\bar{\gamma}_1}) \oplus L^2(\bar{\gamma}_n, \mu_{\bar{\gamma}_n}). \end{aligned}$$

As it was explained above $L_{\Gamma}(X_f) = C(X_f)$ in this case, and $(C(X_f))'_{C_{\Gamma}(X_f)} = C(X_f)$. \square

Example 3.4. Let Y be the cone given by the equation $x^2 + y^2 = z^2$, $Z \subset Y$ be the subset of all points with $z \in J = \{0, 1, 1/2, 1/3, \dots\}$. Then Z is an infinite collection of circles with one limit point $(0, 0, 0)$ added. Let X be a union of three distinct copies of Z . To describe an action of \mathbb{Z} on Z number the circles in the double-cone consecutively by numbers of \mathbb{Z} where the number zero is fixed to the point $(0, 0, 0)$. Consider the discrete group $\Gamma = \mathbb{Z} \oplus \mathbb{Z}_3$, where \mathbb{Z} acts on each circle by an irrational rotation by an angle α_i ($i = 1, 2, \dots$), where $\alpha_i \rightarrow 0$, and where \mathbb{Z}_3 transposes the cones. Then the module $L_{\Gamma}(X)$ is not self-dual since the orbits are all infinite except for the fixed-point.

4. C^* -REFLEXIVITY

4.1. THE METRIC CASE

In this section we would like to understand, in which situations the Hilbert C^* -module $L_{\Gamma}(X)$ is C^* -reflexive over $C_{\Gamma}(X)$. Our previous partial results [5, 18] made us believe that the Hilbert C^* -module $L_{\Gamma}(X)$ is C^* -reflexive in much more general situations beyond the finite orbit case. It turns out that any countably generated module over a wide class of commutative C^* -algebras is C^* -reflexive.

Theorem 4.1. *Let X be a compact metric space. Then any countably generated module over $C(X)$ is C^* -reflexive.*

Proof. The first version of a proof appeared in [9]. Then Trofimov [17] realized that the formulation in [9] was too general and provided a proof for any compact X with a certain property L. While preparing this paper, we understood that the property L of Trofimov for X is the same as the property of X to be a compact Baire space. So, any compact Hausdorff space has this property L, and C^* -reflexivity would have place for any countably generated module over any unital commutative C^* -algebra, which is obviously not true, e.g. for von Neumann algebras [11]. Nevertheless, the main part of Trofimov's proof is correct. But it was overlooked that implicitly the proof used that, for any subset $E \subset X$ and for any point t_0 in the closure of E , there exists a sequence of points $t_n \in E$, which

converges to t_0 . In other words, the topology on X is supposed to possess a countable base of neighborhoods at any point of X . This is not true in general, but if we restrict ourselves to the case of compact *metric* spaces then this is obviously true. Under this additional assumption, Trofimov's proof is correct. \square

Corollary 4.2. *Let X be a compact metric space, and let an action of Γ on X be Lyapunov stable. Then the module $L_\Gamma(X)$ is C^* -reflexive.*

Proof. Since X is metric, the module $L_\Gamma(X)$ is countably generated and the C^* -algebra $C_\Gamma(X)$ is separable, hence its Gelfand spectrum is metrizable. \square

Example 4.3. Let $D = \prod_{k=1}^\infty D_k$, where each D_k is the two-points space with the distance between the two points equal to 2^{-k} , and let $X = J \times D$. Let $G = \oplus_{k=1}^\infty \mathbb{Z}_2$, $G_n = \oplus_{k=1}^n \mathbb{Z}_2$ and $\pi_n : G \rightarrow G_n$, $i_n : G_n \rightarrow G$ be the standard projection and inclusion homomorphisms. Denote their composition by $p_n = i_n \circ \pi_n : G \rightarrow G$. Let α denote the standard action of G on D . Define an action β of G on X by the formula

$$\beta_g \left(\frac{1}{n}, d \right) = \left(\frac{1}{n}, \alpha_{p_n(g)}(d) \right), n \in \mathbb{N} \setminus \{0\}, \text{ and } \beta_g(0, d) = (0, \alpha_g(d)), \text{ where } d \in D.$$

It is easy to see that the following properties hold for this action:

- The orbit of any point of the form $(\frac{1}{n}, d)$ is finite and consists of 2^n elements.
- The orbit of any point of the form $(0, d)$ is infinite.
- The action β is continuous.
- The action β is Lyapunov stable.

It follows from Corollary 4.2 that the module $L_\Gamma(X)$ is C^* -reflexive in this example.

4.2. THE NON-METRIC CASE

After we have clarified, how C^* -reflexivity arises in the metric case, let us pass to the case, when X is non-metric. To begin with, we give an example of a non- C^* -reflexive module $L_\Gamma(X)$.

Example 4.4. Let K be a (non-metrizable) compact space such that $l_2(A)$ is not C^* -reflexive, where $A = C(K)$. That is the case for A being a von Neumann algebra, and one of the most important cases is that of $K = \beta\mathbb{N}$, the Stone–Čech compactification of integers. Consider the compact space $X = K \times S^1$ equipped with the action of \mathbb{Z} by irrational rotation in the second argument:

$$m(y, s) = (y, e^{\alpha\pi m} s), \quad m \in \mathbb{Z}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q}, \quad s \in S^1 \subset \mathbb{C}.$$

This is an isometric action on S^1 and a trivial one on K , hence it is Lyapunov stable. Evidently, the algebra of continuous invariant functions $C_\Gamma(X)$ is $A = C(K)$. By Theorem 3.2, the module $L_\Gamma(X)$ is the set of all functions $\psi : X \rightarrow \mathbb{C}$ such that

- 1) $\psi|_{\bar{\gamma}} \in L^2(\bar{\gamma}, \mu_{\bar{\gamma}})$ for each orbit γ , i.e. $\psi_y(s) = \psi(y, s) \in L^2(S^1)$;
- 2) for any $\varphi \in C(X)$ the function $\langle \psi, \varphi \rangle_L$ is continuous.

Let $\{e_j\}$ be a countable system of orthonormal functions forming an orthonormal basis of $L^2(S^1)$ (e.g. exponents). Then $\{1_A \cdot e_j\}$ is an orthonormal system in $L_\Gamma(X)$:

$$\langle 1_A \cdot e_j, 1_A \cdot e_k \rangle_L(y) = \int_{S^1} 1_A(y) e_j(s) \overline{e_k(s)} 1_A(y) ds = \delta_{jk}.$$

Let us show that the $C(K)$ -linear span of $\{1_A \cdot e_j\}$ is dense in $C(X)$ (hence, in $L_\Gamma(X)$) with respect to the Hilbert module distance. Let $\varphi \in C(X)$. Then for any $\varepsilon > 0$ we can choose a division $\Delta_1, \dots, \Delta_d$ of S^1 such that $\sup_{\Delta_i} (\varphi - f_i) < \frac{\varepsilon}{d}$, where f_i is independent on $s \in S^1$, i.e. actually $f_i \in A$, and $\sup_{\Delta_i} |f_i| \leq 2 \sup_X |\varphi|$. Let χ_i be the indicator function of Δ_i , $i = 1, \dots, d$. Take $\hat{\chi}_i$ to be a \mathbb{C} -linear combination of $\{e_j\}$ such that

$$\|\chi_i - \hat{\chi}_i\|_{L^2(S^1)} < \frac{\varepsilon}{d}, \quad i = 1, \dots, d.$$

Then

$$\hat{\varphi}(y, s) := \sum_{i=1}^d f_i(y) \cdot \hat{\chi}_i(s) \in \text{span}_{C(K)}\{1_A \cdot e_j\}.$$

Let $\psi \in C(X)$, $\|\psi\|_L \leq 1$. Then

$$\begin{aligned} \|\langle \varphi - \hat{\varphi}, \psi \rangle\|_L &= \sup_{y \in K} \left| \int_{S^1} (\varphi(y, s) - \hat{\varphi}(y, s)) \overline{\psi}(y, s) ds \right| \\ &\leq \sup_{y \in K} \left| \int_{S^1} \left(\varphi(y, s) - \sum_{j=1}^d f_j(y) \chi_j(s) \right) \overline{\psi}(y, s) ds \right| \\ &\quad + \sup_{y \in K} \left| \sum_{j=1}^d \int_{S^1} f_j(y) (\chi_j(s) - \hat{\chi}_j(s)) \overline{\psi}(y, s) ds \right| \\ &\leq \sup_{y \in K} \left(\sup_{s \in S^1} \left| \varphi(y, s) - \sum_{j=1}^d f_j(y) \chi_j(s) \right| \right) \left(\int_{S^1} |\psi(y, s)|^2 ds \right)^{1/2} \\ &\quad + \sup_{y \in K} d \cdot \sup_{j=1, \dots, d} |f_j(y)| \cdot \left(\int_{S^1} |\chi_j(s) - \hat{\chi}_j(s)|^2 ds \right)^{1/2} \cdot \left(\int_{S^1} |\psi(y, s)|^2 ds \right)^{1/2} \\ &\leq \sup_{i=1, \dots, d} \sup_{K \times \Delta_i} |\varphi(y, s) - f_i(y)| \cdot \|\psi\|_L + (2 \sup_{x \in X} \|\varphi(x)\|) \cdot d \cdot \frac{\varepsilon}{d} \cdot \|\psi\|_L \\ &< \varepsilon \left(1 + 2 \sup_{x \in X} \|\varphi(x)\| \right). \end{aligned}$$

Thus, $L_\Gamma(X) = l_2(A)$ and is not C^* -reflexive.

Although we are far from obtaining a criterium for C^* -reflexivity, we can give a sufficient condition even in the non-metric case.

Theorem 4.5. *Consider a Lyapunov stable action of Γ on a compact Hausdorff space X , where X is not necessarily metrizable. Suppose, the cardinality of finite orbits is uniformly bounded and the number of closures of infinite orbits is finite. Then $L_\Gamma(X)$ is C^* -reflexive.*

Proof. By the argument in the proof of the first part of Theorem 3.3 (see page 8) finite orbits form a closed invariant subset $X_f \subset X$. The Gelfand spectrum consists of a closed subspace X_f/Γ and a finite number of isolated points corresponding to the closures of infinite orbits. Arguing as in the second part of Theorem 3.3 (see page 10), we reduce this case to the case of pure finite orbits [16]. \square

5. FURTHER EXAMPLES

We want to show by examples that there are other situations beyond the described above in which a well-defined averaging can be found leading to admissible C^* -valued inner products and derived Hilbert C^* -module structures on the corresponding commutative C^* -algebras.

The following example shows that we can have a non-Lyapunov stable action with a good average.

Example 5.1. Let $\Gamma = \mathbb{Z}$. Let X be the direct product $X = J \times S^1$ of the subset

$$J = \{0, 1, 1/2, 1/3, \dots\} \subset \mathbb{R}$$

and the unit circle. Let $\alpha_i \rightarrow \alpha$ be a sequence of irrational numbers, such that α is irrational and α/α_i is irrational for every i . Let the generator of \mathbb{Z} rotate $\{1/i\} \times S^1$ by α_i , and the limit circle $\{0\} \times S^1$ by α . Clearly we have 1') and 2') in this case.

The next example demonstrates that in the case of presence of infinite orbits, uniform continuity is not sufficient for continuity of the average.

Example 5.2. [18, Example 25] Let $X \subset \mathbb{R}^3$ consist of two circles

$$S_{\pm} : \begin{cases} x = \cos 2\pi t \\ y = \sin 2\pi t \\ z = \pm 1, \end{cases} \quad t \in (-\infty, +\infty),$$

and of a non-uniform spiral

$$\Sigma : \begin{cases} x = \cos 2\pi\tau \\ y = \sin 2\pi\tau \\ z = \frac{2}{\pi} \cdot \arctan \tau, \end{cases} \quad (-\infty, +\infty).$$

Let the generator g of $\Gamma = \mathbb{Z}$ act on all three components by

$$t \mapsto t + \alpha, \quad \tau \mapsto \tau + \alpha,$$

where α is a positive irrational number. Then the isotropy group of each point of X is trivial. Hence, the condition of uniform continuity holds automatically.

Let $\varphi : X \rightarrow \mathbb{R}$ be the restriction of the function $\mathbb{R}^3 \ni (x, y, z) \mapsto z$ onto X , then the function φ_x on \mathbb{Z} has the following form: if $x \in S_{\pm}$ then $\varphi_x = \pm 1$; if $x \in \Sigma$ then φ_x is a function on \mathbb{Z} such that $\varphi_x(n) \in [-1, 1]$ for any $n \in \mathbb{Z}$ and $\lim_{n \rightarrow \pm\infty} \varphi_x(n) = \pm 1$. So, φ_x is in general not almost periodic and we cannot average it using our definition. Nevertheless, we can average it using the amenability of the group \mathbb{Z} . In this case we get

$E_{\Gamma}(\varphi_x) = \begin{cases} \pm 1 & \text{for } x \in S_{\pm} \\ 0 & \text{for } x \in \Sigma \end{cases}$. Thus we see that $E_{\Gamma}(\varphi_x)$ is not continuous with respect to $x \in X$.

Example 5.3. In the previous example let us identify the two circles, S_+ and S_- . Then X would consist of the spiral Σ and of the circle S . Still, the function φ_x on $\Gamma = \mathbb{Z}$ need not be almost periodic, but there is an almost periodic function ρ on \mathbb{Z} such that for any $\varepsilon > 0$ there is finite subset $F \subset \mathbb{Z}$ such that $\|\varphi_x - \rho\| < \varepsilon$ on $\mathbb{Z} \setminus F$. This makes it possible to define an average $E_{\Gamma}(\varphi)$ by $E_{\Gamma}(\varphi_x) = M(\rho)$. And it is easy to see that, this time, $E_{\Gamma}(\varphi_x)$ is continuous with respect to $x \in X$.

Example 5.4. Our next example is a modification of Example 4.4. Let $Y = \mathbb{N} \times \mathbb{S}^1$, $X = \beta Y$ its Stone–Čech compactification. Let $\Gamma = \mathbb{Z}$ act on Y by rotating each circle by the irrational angle α . This action canonically extends to an action on X .

Let $s \in \mathbb{S}^1$. Then the inclusion $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{S}^1$, $n \mapsto (n, s)$, canonically extends to a map $s_* : \beta\mathbb{N} \rightarrow X$ and $m(s_*(x)) = (s \cdot e^{im\alpha})_*(x)$ for any $x \in \beta\mathbb{N}$ and any $m \in \Gamma = \mathbb{Z}$.

Let $\varphi \in C(X)$. Since $C(X) = C_b(Y)$ (continuous functions on X are canonically identified with bounded continuous functions on Y), φ can be identified with a uniformly bounded sequence $(\varphi^{(1)}, \varphi^{(2)}, \dots)$ of continuous functions on \mathbb{S}^1 . Let $x \in \beta\mathbb{N}$. Then

$$\varphi_{s_*(x)}(m) = \varphi((s \cdot e^{im\alpha})_*(x)),$$

where $m \in \mathbb{Z}$.

Let \mathcal{U}_x be an ultrafilter on \mathbb{N} , which corresponds to the point $x \in \beta\mathbb{N}$. Then

$$\varphi_{s_*(x)} = \lim_{\mathcal{U}_x} (\varphi^{(n)}(s))_{n=1}^\infty,$$

where the limit of the sequence $(\varphi^{(n)}(s))_{n=1}^\infty$ is taken over \mathcal{U}_x , hence

$$\varphi_{s_*(x)}(m) = \lim_{\mathcal{U}_x} (\varphi^{(n)}((s \cdot e^{im\alpha})_*(x)))_{n=1}^\infty.$$

Take $\varphi^{(n)}(s) = e^{ins}$. Then $\varphi \in C_b(Y) = C(X)$. Then

$$\varphi_{1_*(x)}(m) = \lim_{\mathcal{U}_x} (e^{inm\alpha})_{n=1}^\infty.$$

Let \mathcal{U} be an ultrafilter on \mathbb{N} such that $\lim_{\mathcal{U}} (e^{in\lambda})_{n=1}^\infty = 0$ for any $\lambda \in (0, 2\pi)$, and let $x_0 \in \beta\mathbb{N}$ be the point that corresponds to \mathcal{U} . Then

$$\varphi_{1_*(x_0)}(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases}$$

Thus, for the point $y = 1_*(x_0) \in X$ and for the function $\varphi \in C(X)$ we see that the function φ_y is not almost periodic on \mathbb{Z} .

Nevertheless, there is a ‘good’ averaging in this example. Since any continuous function φ on X is a uniformly bounded sequence of functions $\varphi^{(n)}$, $n \in \mathbb{N}$, on \mathbb{S}^1 , it is easy to see that $C_\Gamma(X) \cong C_b(\mathbb{N})$, and one can define $E_\Gamma(\varphi)$ by the formula $(E_\Gamma(\varphi))_n = \int_{\mathbb{S}^1} \varphi^{(n)}(s) ds$.

These examples show that a good averaging (and an inner product with values in $C_\Gamma(X)$) can be defined in a wider class than Lyapunov stable actions. On the other hand, as the last two examples show, a good averaging, when exists, may give rise to a degenerate inner product.

6. APPENDIX

In this section we will prove the following assertion on the uniqueness of invariant regular measures:

Lemma 6.1. *Suppose, a discrete group Γ acts on a compact Hausdorff space X in such a way that the orbit γ of an element $a \in X$ is dense in X . If the action is Lyapunov stable, then X carries not more than one invariant regular measure.*

In fact the assertion follows from [2, Ch. VII, § 1, Problem 14]. More precisely, denote the closure $\overline{\gamma}$ of an orbit γ by T . We can simplify the idea of the argument of [2, Ch. VII, § 1, Problem 14] because T is a compact Hausdorff space. For any subset $A \subset T$ and any $B \subset T$ with interior points denote by $(A : B)$ the minimal cardinality of covers of A formed by sets gB , $g \in \Gamma$. Density of γ in T and Lyapunov stability imply the finiteness of this number, or, more precisely, that such a cover exists. To prove this, we will show that for any open set B we have $\Gamma(B) = T$. Suppose the opposite: $x \notin \Gamma(B)$. Then Γx is not dense in X . A contradiction to Lemma 3.1.

In the uniform structure of T there exists a fundamental sub-system S formed by invariant sets in $T \times T$ (under the diagonal action of Γ). Note, if $\Gamma(\mathbb{V}) \subset \mathbb{U}$ by Lyapunov stability, then $\cup_{g \in \Gamma} g(\mathbb{V})$ is an invariant neighborhood of the diagonal, containing \mathbb{V} and contained in \mathbb{U} .

If $C \subset T$ is a third relatively compact set with interior points, then $(A : C) \leq (A : B) \cdot (B : C)$.

Recall the following notation. If \mathbb{V} and \mathbb{W} are subsets of $T \times T$, then $\mathbb{V}\mathbb{W} \subset T \times T$ is formed by all pairs (x, y) such that there exists an element $z \in T$ with $(x, z) \in \mathbb{V}$ and $(z, y) \in \mathbb{W}$. If $K \subset T$ is an arbitrary set, then

$$\mathbb{V}(K) := \{y \in T \mid (x, y) \in \mathbb{V} \text{ for some } x \in K\}.$$

If $K = \{a\}$, we write $\mathbb{V}(a)$.

Suppose, $K \subset T$ is a compact subset, $L \supset K$ is a open set, and $\mathbb{V} \in S$ is an invariant symmetric open neighborhood of the diagonal, such that $\mathbb{V}(K) \subset L$, and $\mathbb{W} \in S$ is a closed invariant symmetric neighborhood of the diagonal such that $\mathbb{W} \subset \mathbb{V}$. Let \mathbb{U} be a symmetric invariant set containing the diagonal such that $\mathbb{U}\mathbb{W} \subset \mathbb{V}$ and $\mathbb{W}\mathbb{U} \subset \mathbb{V}$. Then for any invariant (regular) positive measure ν one has

$$(10) \quad (\mathbb{W}(a) : \mathbb{U}(a)) \cdot \nu(K) \leq (L : \mathbb{U}(a)) \cdot \nu(\mathbb{V}(a)),$$

$$(11) \quad (K : \mathbb{U}(a)) \cdot \nu(\mathbb{W}(a)) \leq (\mathbb{V}(a) : \mathbb{U}(a)) \cdot \nu(L).$$

Indeed, let $L = \bigcup_{i=1}^{(L:\mathbb{U}(a))} g_i \mathbb{U}(a)$, $g_i \in \Gamma$. Then each $x \in K$ belongs at least to $(\mathbb{W}(a) : \mathbb{U}(a))$ sets from the collection $\{g_i \mathbb{V}(a)\}$. Indeed, $\mathbb{W}(x)$ is covered by $g_i \mathbb{U}(a)$. Hence, its number (number of those of them, which really intersect $\mathbb{W}(x)$) is greater-equal to $(\mathbb{W}(x) : \mathbb{U}(a)) = (\mathbb{W}(a) : \mathbb{U}(a))$. The last equality follows from the density of the orbit, Lyapunov property and the invariance of \mathbb{W} : $\mathbb{W}(gx) = g\mathbb{W}(x)$. It remains to show that if $g_i \mathbb{U}(a) \cap \mathbb{W}(x) \neq \emptyset$, then $x \in g_i \mathbb{V}(a)$. In this case let $g_i s \in g_i \mathbb{U}(a) \cap \mathbb{W}(x)$ for some $s \in T$, then $(g_i a, g_i s) \in \mathbb{U}$, $(x, g_i s) \in \mathbb{W}$. Since the sets are symmetric and $\mathbb{W}\mathbb{U} \subset \mathbb{V}$, we have $(x, g_i a) \in \mathbb{V}$. Then $(a, g_i^{-1}x) \in \mathbb{V}$, $g_i^{-1}x \in \mathbb{V}(a)$, $x \in g_i \mathbb{V}(a)$. Thus,

$$\nu(K) \leq \frac{1}{(\mathbb{W}(a) : \mathbb{U}(a))} \sum_{i=1}^{(L:\mathbb{U}(a))} \nu(g_i \mathbb{U}(a)).$$

Since ν is invariant, we obtain (10). To obtain (11) in a similar way, we will show that each $y \in L$ belongs to not more than $(\mathbb{V}(a) : \mathbb{U}(a))$ sets of the form $g_i \mathbb{W}(a)$. Indeed, if $y \in g_i \mathbb{W}(a)$, then $y = g_i s$ and $(a, s) \in \mathbb{W}$ for some $s \in T$, as well as $(g_i a, g_i s)$. Let $(a, t) \in \mathbb{U}$. Since $\mathbb{W}\mathbb{U} \subset \mathbb{V}$, $(s, t) \in \mathbb{V}$, $t \in \mathbb{V}(s)$, $\mathbb{U}(a) \subset \mathbb{V}(s)$, $g_i \mathbb{U}(a) \subset g_i \mathbb{V}(s) = \mathbb{V}(y)$. So, $g_i \mathbb{U}(a)$ ($i = 1, \dots, (L : \mathbb{U}(a))$) form a minimal cover of L while a part of this cover is inside $\mathbb{V}(y) \subset L$. Thus the cardinality of this part is lower-equal to $(\mathbb{V}(y) : \mathbb{U}(a)) =$

$(\mathbb{V}(a) : \mathbb{U}(a))$. Hence,

$$\nu(L) \geq \frac{1}{(\mathbb{V}(a) : \mathbb{U}(a))} \cdot \sum_{i=1}^{(L:\mathbb{U}(a))} \nu(g_i \mathbb{W}(a)) = \frac{(L : \mathbb{U}(a))\nu(\mathbb{W}(a))}{(\mathbb{V}(a) : \mathbb{U}(a))} \geq \frac{(K : \mathbb{U}(a))\nu(\mathbb{W}(a))}{(\mathbb{V}(a) : \mathbb{U}(a))}$$

and we obtain (11).

Suppose, $A, A_0 \subset T$ are non-empty open set, $K \subset T$ is a compact set. Choose a fundamental system of open invariant symmetric neighborhoods \mathbb{U}_α of the diagonal Δ indexed by a net \mathcal{A} . Put

$$\lambda_\alpha(A) := \frac{(A : \mathbb{U}_\alpha(a))}{(A_0 : \mathbb{U}_\alpha(a))}, \quad \lambda(A) := \lim_{\mathcal{A}} \lambda_\alpha(A) \quad \text{and} \quad \lambda'(K) := \inf \lambda(B),$$

where B runs over the set of all relatively compact open neighborhoods of K . Then from (10) and (11) we obtain

$$(12) \quad \lambda'(\mathbb{W}(a)) \cdot \nu(K) \leq \lambda(L) \cdot \nu(\mathbb{W}(a)), \quad \lambda'(K) \cdot \nu(\mathbb{W}(a)) \leq \lambda'(\mathbb{W}(a)) \cdot \nu(L)$$

for \mathbb{W} sufficiently close to the diagonal to have $\mathbb{W}(K) \subset L$. Indeed, choose open symmetric invariant neighborhoods of \mathbb{W} inside its sufficiently small \mathbb{U}_α -neighborhood: \mathbb{W}_α and \mathbb{V}_α (i.e. $\mathbb{W}_\alpha \subset \mathbb{U}_\alpha \mathbb{W}$, $\mathbb{V}_\alpha \subset \mathbb{U}_\alpha \mathbb{W}$) such that $\overline{\mathbb{W}_\alpha} \subset \mathbb{V}_\alpha$ and still $\mathbb{V}_\alpha(K) \subset L$. Then choose $\beta_0 = \beta_0(\alpha)$ such that $U_\beta \overline{\mathbb{W}_\alpha} \subset \mathbb{V}_\alpha$, for all $\beta > \beta_0$. Take in (10) $\mathbb{W} = \overline{\mathbb{W}_\alpha}$, $\mathbb{V} = \mathbb{V}_\alpha$, $\mathbb{U}(a) := \mathbb{U}_\beta(a)$, and divide both parts by $(A_0 : \mathbb{U}_\beta(a))$:

$$\begin{aligned} \frac{(\overline{\mathbb{W}_\alpha}(a) : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(K) &\leq \frac{(L : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(\mathbb{V}_\alpha(a)), \\ \frac{(\mathbb{W}_\alpha(a) : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(K) &\leq \frac{(L : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(\mathbb{V}_\alpha(a)). \end{aligned}$$

Passing to the limit over $\beta \in \mathcal{A}$ we obtain

$$\lambda(\mathbb{W}_\alpha(a)) \cdot \nu(K) \leq \lambda(L) \cdot \nu(\mathbb{V}_\alpha(a))$$

for any $\alpha \in \mathcal{A}$. Passing to the limit over $\alpha \in \mathcal{A}$ we get

$$\lambda'(\mathbb{W}(a)) \cdot \nu(K) \leq \lambda(L) \cdot \nu(\mathbb{W}(a)).$$

To obtain the second inequality in (12) choose a sufficiently small open neighborhood K_α of K to have $\overline{K_\alpha} \subset L$ and add to the above restrictions the following one: $\mathbb{V}_\alpha(\overline{K_\alpha}) \subset L$. Then from (11) we have

$$\begin{aligned} \frac{(\overline{K_\alpha} : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(\mathbb{W}(a)) &\leq \frac{(V_\alpha(a) : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(L), \\ \frac{(K_\alpha : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(\mathbb{W}(a)) &\leq \frac{(\mathbb{V}_\alpha(a) : \mathbb{U}_\beta(a))}{(A_0 : \mathbb{U}_\beta(a))} \nu(L). \end{aligned}$$

Passing to the limit over $\beta \in \mathcal{A}$ the inequality

$$\lambda(K_\alpha) \cdot \nu(\mathbb{W}(a)) \leq \lambda(\mathbb{V}_\alpha(a)) \cdot \nu(L)$$

holds for any $\alpha \in \mathcal{A}$. Now passing to the limit over $\alpha \in \mathcal{A}$ we get

$$\lambda'(K) \cdot \nu(\mathbb{W}(a)) \leq \lambda'(\mathbb{W}(a)) \cdot \nu(L).$$

In particular, if $K_1 \subset T$ and $K_2 \subset T$ are compact subsets, and $L_1 \supset K_1$ and $L_2 \supset K_2$ are relatively compact open sets, then (12) implies

$$\frac{\nu(K_2)}{\lambda(L_2)} \leq \frac{\nu(\mathbb{W}(a))}{\lambda'(\mathbb{W}(a))} \leq \frac{\nu(L_1)}{\lambda'(K_1)}, \quad \lambda'(K_1) \cdot \nu(K_2) \leq \lambda(L_2) \cdot \nu(L_1).$$

Passing to the infimum over $L_2 \supset K_2$ (by the definition of λ') and over $L_1 \supset K_1$ (by the regularity of ν) we obtain $\lambda'(K_1) \cdot \nu(K_2) \leq \lambda'(K_2) \cdot \nu(K_1)$. Transposing K_1 and K_2 we have $\lambda'(K_2) \cdot \nu(K_1) \leq \lambda'(K_1) \cdot \nu(K_2)$. Hence, $\lambda'(K_1) \cdot \nu(K_2) = \lambda'(K_2) \cdot \nu(K_1)$. This uniquely determines ν (up to a constant multiplier).

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